

ON THE JOINT SPECTRA OF THE TWO DIMENSIONAL LIE ALGEBRA OF OPERATORS IN HILBERT SPACES

ENRICO BOASSO

ABSTRACT. We consider the complex solvable non-commutative two dimensional Lie algebra L , $L = \langle y \rangle \oplus \langle x \rangle$, with Lie bracket $[x, y] = y$, as linear bounded operators acting on a complex Hilbert space H . Under the assumption $R(y)$ closed, we reduce the computation of the joint spectra $Sp(L, E)$, $\sigma_{\delta,k}(L, E)$ and $\sigma_{\pi,k}(L, E)$, $k = 0, 1, 2$, to the computation of the spectrum, the approximate point spectrum, and the approximate compression spectrum of a single operator. Besides, we also study the case $y^2 = 0$, and we apply our results to the case H finite dimensional.

1. INTRODUCTION

In [1] we introduced a joint spectrum for complex solvable finite dimensional Lie algebras of operators acting on a Banach space E . If L is such an algebra, and $Sp(L, E)$ denotes its joint spectrum, $Sp(L, E)$ is a compact non empty subset of L^* , which also satisfies the projection property for ideals, i. e., if I is an ideal of L and $\Pi: L^* \rightarrow I^*$ denotes the restriction map, then $Sp(I, E) = \Pi(Sp(L, E))$. In addition, when L is a commutative algebra, $Sp(L, E)$ reduces to the Taylor joint spectrum, see [5]. Moreover, in [2] we extended Słodkowski joint spectra $\sigma_{\delta,k}$ and $\sigma_{\pi,k}$ to the case under consideration and we proved the usual spectral properties: they are compact non empty subsets of L^* and the projection property for ideals still holds.

In this paper we consider the complex solvable non-commutative two dimensional Lie algebra L , $L = \langle y \rangle \oplus \langle x \rangle$, with Lie bracket $[x, y] = y$, as bounded linear operators acting on a complex Hilbert space H , and we compute the joint spectra $Sp(L, H)$, $\sigma_{\delta,k}(L, H)$ and $\sigma_{\pi,k}(L, H)$, for $k = 0, 1, 2$, when $R(y)$ is a closed subspace of H . Besides, by means of an homological argument, we reduce the computation of these spectra to the one dimensional case. We prove that these joint spectra are determined by the spectrum, the approximate point spectrum, and the approximate compression spectrum of x in $Ker(y)$ and \bar{x} in $H/R(y)$, where \bar{x} is the quotient map associated to x , $(R(y))$ and $Ker(y)$ are invariant subspaces for the operator x .

In addition, we consider the case $y^2 = 0$ (it easy to see that y is a nilpotent operator), and we obtain a relation between the spectrum of x in $R(y)$ and a subset of the spectrum of \bar{x} in $H/R(y)$, which give us a more precise characterization of the joint spectrum $Sp(L, E)$. Finally, we apply our computation to the case H finite dimensional.

The paper is organized as follows. In Section 2 we review several definitions and results of [1] and [2]. In Section 3 we prove our main theorems and, in Section 4, we consider the case $y^2 = 0$ and the finite dimensional case.

2. PRELIMINARIES

In this section we briefly recall the definitions of the joint spectra $Sp(L, H)$, $\sigma_{\delta, k}(L, H)$ and $\sigma_{\pi}(L, H)$, $k = 0, 1, 2$. We restrict ourselves to the case under consideration. For a complete account of the definitions and mean properties of these joint spectra, see [1] and [2].

From now on, let L be the complex solvable two dimensional Lie algebra, $L = \langle y \rangle \oplus \langle x \rangle$, with Lie bracket $[x, y] = y$, which acts as right continuous linear operators on a Hilbert space H , i. e., L is a Lie subalgebra of $\mathcal{L}(H)^{op}$, where $\mathcal{L}(H)$ is the algebra of all bounded linear operators defined on H , and where $\mathcal{L}(H)^{op}$ means that we consider $\mathcal{L}(H)$ with its opposite product. We observe that any complex solvable non-commutative two dimensional Lie algebra may be presented in the above form.

If f is a character of L , we consider the chain complex $(H \otimes \wedge L, d(f))$, where $\wedge L$ denotes the exterior algebra of L , and $d(f)$ is the following map:

$$\begin{aligned} d_{p-1}(f): H \otimes \wedge^p L &\rightarrow H \otimes \wedge^{p-1} L, \\ d_0(f)(a \langle y \rangle) &= y(a), \quad d_0(f)(b \langle x \rangle) = (x - f(x))(b), \\ d_1(f)(c \langle yx \rangle) &= -(x - 1 - f(x))(c) \langle y \rangle + y(c) \langle x \rangle. \end{aligned}$$

Let $H_*(H \otimes \wedge L, d(f))$ denote the homology of the complex $(H \otimes \wedge L, d(f))$, we now state our first definition.

Definition 2.1. *With H , L and f as above, the set $\{f \in L^*: f(L^2) = 0, H_*(H \otimes \wedge L, d(f)) \neq 0\}$, is the joint spectrum of L acting on H , and it is denoted by $Sp(L, H)$.*

As a consequence of the results of [1], we have that $Sp(L, H)$ is a compact non empty subset of L^* . Besides, as a standard calculation shows that the equality $y = [x, y]^{op} = [y, x]$ implies $ny^n = [y^n, x] = [x, y^n]^{op}$, we have that y is a nilpotent operator. Thus, $Sp(\langle y \rangle) = 0$, and by the projection property, if f belongs to $Sp(L, H)$, as $\langle y \rangle = L^2$ is an ideal of L , $f(y) = 0$.

Now, let us consider the basis of L , A , defined by, $A = \{y, x\}$, and B , the basis of L^* dual of A . If we consider $Sp(L, H)$ in terms of the above basis, and we denote it by $Sp((y, x), H)$, i. e., $Sp((y, x), H) = \{(f(y), f(x)): f \in Sp(L, H)\}$, we have that, $Sp((y, x), H) = \{(0, f(x)): f \in Sp(L, H)\}$.

In addition, the complex $(H \otimes \wedge L, d(f))$ may be written in the following way,

$$\begin{aligned} 0 \rightarrow H &\xrightarrow{d_1} H \oplus H \xrightarrow{d_0} H \rightarrow 0, \\ d_0 &= \begin{pmatrix} y & x - \lambda \end{pmatrix}, \quad d_1 = \begin{pmatrix} -(x - 1 - \lambda) \\ y \end{pmatrix}, \end{aligned}$$

where $\lambda = f(x)$. We denote this chain complex by $(C, d(\lambda))$. Thus, as $(0, \lambda) \in Sp((y, x), H)$ if and only if $f \in Sp(L, H)$, where $\lambda = f(x)$, to compute the latter

is equivalent to compute the former, and to study the exactness of the chain complex $(H \otimes \wedge L, d(f))$ is equivalent to study the exactness of $(C, d(\lambda))$.

With regard to the joint spectra $\sigma_{\delta,k}(L, H)$ and $\sigma_{\pi,k}(L, H)$, $k = 0, 1, 2$, we review, for the case under consideration, the definition of them given in [2]. If $p = 0, 1, 2$, let $\Sigma_p(L, H)$ be the set $\Sigma_p(L, H) = \{f \in L^*: f(L^2) = 0, H_p((H \otimes \wedge L, d(f))) \neq 0\}$. We now state our second definition.

Definition 2.2. *With H, L and f as above,*

$$\sigma_{\delta,k}(L, H) = \bigcup_{0 \leq p \leq k} \Sigma_p(L, H),$$

$$\sigma_{\pi,k}(L, H) = \bigcup_{k \leq p \leq 2} \Sigma_p(L, H) \bigcup \{f \in L^*: f(L^2) = 0, R(d_k(f)) \text{ is not closed}\},$$

where $0 \leq k \leq 2$.

We observe that $Sp(L, H) = \sigma_{\delta,2}(L, H) = \sigma_{\pi,0}(L, H)$. Besides, as we have said, these joint spectra are compact non empty subsets of L^* . In addition, as in the case of the joint spectrum $Sp(L, H)$, we consider the joint spectra $\sigma_{\delta,k}(L, H)$ and $\sigma_{\pi,k}(L, H)$ in terms of the basis A and B . As these joint spectra are subsets of $Sp(L, H)$, we have that $\sigma_{\delta,k}((y, x), H) = \{(0, f(x)): f \in \sigma_{\delta,k}(L, H)\}$, and $\sigma_{\pi,k}((y, x), H) = \{(0, f(x)): f \in \sigma_{\pi,k}(L, H)\}$, where $k = 0, 1, 2$.

Moreover, as in the case of the joint spectrum $Sp(L, H)$, to compute $\sigma_{\delta,k}(L, H)$ and $\sigma_{\pi,k}(L, H)$, $0 \leq k \leq 2$, is equivalent to compute these joint spectra in terms of the basis A and B . Finally, to compute the latter joint spectra it is enough to study the complex $(C, d(\lambda))$, and to consider the corresponding properties involved in the definition of $\sigma_{\delta,k}(L, H)$ and $\sigma_{\pi,k}(L, H)$, $0 \leq k \leq 2$, for it.

3. THE MAIN RESULT

We begin with the characterization of $Sp(L, H)$. Indeed, we consider $Sp((y, x), H)$, and by means of an homological argument we reduce its computation to the case of a single operator.

Let us consider the chain complex $(\overline{C}, \overline{d})$,

$$0 \rightarrow H \xrightarrow{\overline{d}=y} H \rightarrow 0.$$

Then an easy calculation shows that we have a short exact sequence of chain complex of the form,

$$0 \rightarrow (\overline{C}, \overline{d}) \xrightarrow{i} (C, d(\lambda)) \xrightarrow{p} (\overline{C}, \overline{d}) \rightarrow 0,$$

where $(i_j)_{(0 \leq j \leq 2)}$ and $(p_j)_{(0 \leq j \leq 2)}$ are the following maps: $i_2 = 0$, $i_1 = I_H \oplus 0$, $i_0 = I_H$, and $p_2 = I_H$, $p_1 = 0 \oplus I_H$, $p_0 = 0$.

Thus, by [4, Chapter II, Section 4, Theorem 4.1], and the fact that p is a map of degree -1 , we have a long exact sequence of homology spaces of the form,

$$\rightarrow H_q(C, d(\lambda)) \xrightarrow{p_{q*}} H_{q-1}(\overline{C}, \overline{d}) \xrightarrow{\partial_{q-1}} H_{q-1}(\overline{C}, \overline{d}) \xrightarrow{i_{q-1*}} H_{q-1}(C, d(\lambda)) \rightarrow.$$

We observe that $H_1(\overline{C}, \overline{d}) = \text{Ker}(y)$ and that $H_0(\overline{C}, \overline{d}) = H/R(y)$. Moreover, as $[x, y]^{op} = y$, we have that $x(R(y)) \subseteq R(y)$ and that $x(\text{Ker}(y)) \subseteq \text{Ker}(y)$.

Then, by [4, Chapter II, Section 4, Theorem 4.1], ∂_q , $q = 0, 1$, are the following maps: $\partial_0([a]) = [(x - \lambda)(a)] = (\bar{x} - \lambda)[a]$, and $\partial_1(b) = -(x - \lambda - 1)(b)$, where $\bar{x}: H/R(y) \rightarrow H/R(y)$ is the map obtained by passing x to the quotient space $H/R(y)$. We now give our characterization of $Sp(L, H)$.

Proposition 3.1. *Let L be the complex solvable non-commutative two dimensional Lie algebra $L = \langle y \rangle \oplus \langle x \rangle$, with Lie bracket $[x, y] = y$, which acts as right continuous linear operators on a complex Hilbert space H . If $R(y)$ is a closed subspace of H and we consider $Sp(L, H)$ in terms of the basis $\{y, x\}$ of L and the basis of L^* dual of the latter, then we have,*

$$Sp((y, x), H) = \{0\} \times Sp(x - 1, Ker(y)) \cup \{0\} \times Sp(\bar{x}, H/R(y)).$$

In addition, we have:

- (i) $H_0(C, d(\lambda)) = 0$ iff $\bar{x} - \lambda: H/R(y) \rightarrow H/R(y)$ is a surjective map,
- (ii) $H_2(C, d(\lambda)) = 0$ iff $x - 1 - \lambda: Ker(y) \rightarrow Ker(y)$ is an injective map,
- (iii) $H_1(C, d(\lambda)) = 0$ iff $\bar{x} - 1 - \lambda$ is injective, and $x - \lambda - 1$ is surjective.

Proof. It is a consequence of the long exact sequence of homology spaces and the form of the maps ∂_j , $j = 0, 1$. \square

In order to characterize the joint spectra $\sigma_{\pi, k}(L, H)$, we recall the notion of approximate point spectrum of an operator T : λ is in the approximate point spectrum of T , which we denote by $\Pi(T)$, if there exists a sequence of unit vectors, $(x_n)_{n \in \mathbb{N}}$, $x_n \in H$, $\|x_n\| = 1$, such that $(T - \lambda)(x_n) \rightarrow 0$ ($n \rightarrow \infty$). An easy calculation shows that $\lambda \notin \Pi(T)$ if and only if $Ker(T - \lambda) = 0$ and $R(T - \lambda)$ is closed in H .

We now consider the spectrum $\sigma_{\pi, 2}((y, x), H)$. We observe that, as $[x, y]^{op} = y$, $(x - 1)(Ker(y)) \subseteq Ker(y)$. Then, we may consider $\Pi(x - 1, Ker(y))$. Indeed, we shall see that $\sigma_{\pi, 2}((y, x), H) = \{0\} \times \Pi(x - 1, Ker(y))$.

To prove the last assertion we proceed as follows. By Definition 2, we have that $\sigma_{\pi, 2}^c = \{(0, \lambda): H_2(C, d(\lambda)) = 0 \text{ and } R(d_1(\lambda)) \text{ is closed}\}$. However, by the definition of $d_1(\lambda)$ and $H_2(C, d(\lambda))$, $H_2(C, d(\lambda)) = Ker(x - 1 - \lambda) \cap Ker(y)$. Then, $H_2(C, d(\lambda)) = 0$ is equivalent to $Ker(x - 1 - \lambda|_{Ker(y)}) = 0$. Thus, in order to conclude with our assertion, it is enough to see that the fact $R(x - 1 - \lambda|_{Ker(y)})$ is closed, is equivalent to $R(d_1(\lambda))$ is closed.

Indeed, if $(a_n)_{n \in \mathbb{N}}$ is a sequence in $Ker(y)$ such that $(x - 1 - \lambda)(a_n) \rightarrow b \in Ker(y)$ ($n \rightarrow \infty$), we have that, $d_1(\lambda)(a_n) \rightarrow (-b, 0)$ ($n \rightarrow \infty$). If $R(d_1(\lambda))$ is closed, there is a z in H such that $d_1(\lambda)(z) = (-b, 0)$, i.e., $-(x - 1 - \lambda)(z) = -b$ and $y(z) = 0$. Thus, $z \in Ker(y)$ and $R((x - 1 - \lambda)|_{Ker(y)})$ is closed.

On the other hand, if $R((x - 1 - \lambda)|_{Ker(y)})$ is closed, let us consider a sequence $(z_n)_{n \in \mathbb{N}}$, $z_n \in H$, such that $d_1(\lambda)(z_n) \rightarrow (w_1, w_2) \in H \oplus H$ ($n \rightarrow \infty$). We decompose H as the orthogonal direct sum of $Ker(y)$ and $Ker(y)^\perp$, $H = Ker(Y) \oplus Ker(y)^\perp$. Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be sequences in $Ker(y)$ and $Ker(y)^\perp$, respectively, such that $z_n = a_n + b_n$. Then,

$$\begin{aligned} d_1(\lambda) &= d_1(\lambda)(a_n) + d_1(\lambda)(b_n) \\ &= (-(x - 1 - \lambda)(a_n), 0) + (-(x - 1 - \lambda)(b_n), \bar{y}(b_n)), \end{aligned}$$

where $\bar{y}: Ker(y)^\perp \rightarrow R(y)$ is the restriction of y to $Ker(y)^\perp$. We observe that, as $R(y)$ is a closed subspace of H , \bar{y} is a topological homeomorphism. Besides, as $\bar{y}(b_n) \rightarrow w_2$ ($n \rightarrow \infty$), there exists a $z_2 \in Ker(y)^\perp$ such that $b_n \rightarrow z_2$ ($n \rightarrow \infty$), and $\bar{y}(z_2) = w_2$. Then, $-(x-1-\lambda)(b_n) \rightarrow -(x-1-\lambda)(z_2)$ ($n \rightarrow \infty$), and $-(x-1-\lambda)(a_n) \rightarrow w_1 + (x-1-\lambda)(z_2)$ ($n \rightarrow \infty$). As $(a_n)_{n \in \mathbb{N}}$ is a sequence in $Ker(y)$, and $R(x-1-\lambda|_{Ker(y)})$ is closed, there is a $z_1 \in Ker(y)$ such that $w_1 + (x-1-\lambda)(z_2) = -(x-1-\lambda)(z_1)$. Thus, $(w_1, w_2) = d_1(\lambda)(z_1 + z_2)$, equivalently, $R(d_1(\lambda))$ is a closed subspace of $H \oplus H$.

With regard to $\sigma_{\pi,1}((y, x), H)$, we have, by Definition 2.2, that,

$$\sigma_{\pi,1}((y, x), H)^c = \{(0, \lambda): H_i(C, d(\lambda)) = 0, i = 1, 2, \text{ and } R(d_0(\lambda)) \text{ is closed}\},$$

which, by Proposition 1, is equivalent to the following conditions:

- (i) $x-1-\lambda: Ker(y) \rightarrow Ker(y)$ is an isomorphic map,
- (ii) $\bar{x}-\lambda: H/R(y) \rightarrow H/R(y)$ is an injective map,
- (iii) $R(d_0(\lambda))$ is closed.

We shall see that $\sigma_{\pi,1}((y, x), H) = Sp(x-1, Ker(y)) \cup \Pi(\bar{x}, H/R(y))$.

Indeed, it is clear that condition (i) is equivalent to $\lambda \notin Sp(x-1, Ker(y))$. Then, it is enough to see that condition (ii)-(iii) are equivalent to $\lambda \notin \Pi(\bar{x}, H/R(y))$. However, by (ii), it suffices to verify that the fact $R(d_0)(\lambda)$ is closed is equivalent to $R(\bar{x}-\lambda)$ is closed. Now, as the quotient map, $\Pi: H \rightarrow H/R(y)$, is an identification, by [3, Chapter II, Section 6, Lemma 6.1], $R = R(\bar{x}-\lambda) = \Pi(R(x-\lambda))$ is closed in $H/R(y)$ if and only if $\Pi^{-1}(R) = R(x-\lambda) + R(y) = R(d_0(\lambda))$ is closed in H .

In order to study the joint spectra $\sigma_{\delta,k}(L, H)$, $k = 0, 1, 2$, we recall the definition of the approximate compression Spectrum of an operator T in H : λ is in the approximate compression spectrum of T , which we denote by $\Pi C(T)$, if there exists a sequence of unit vectors in H , $(x_n)_{n \in \mathbb{N}}$, $x_n \in H$, $\|x_n\| = 1$, such that $(T-\lambda)^*(x_n) \rightarrow 0$ ($n \rightarrow \infty$), i. e., $\Pi C(T) = \Pi(T^*)$. Besides, an easy calculation shows that λ does not belong to $\Pi(T)$ if and only if $(T-\lambda)$ is a surjective map.

We now consider the joint spectra $\sigma_{\delta,o}((y, x), H)$. However, by Definition 2.2, Proposition 3.1 and the previous considerations about the approximate compression spectrum, it is clear that $\sigma_{\delta,k}((y, x), H) = \{0\} \times \Pi C(\bar{x}, H/R(y))$.

With regards to $\sigma_{\delta,1}((y, x), H)$, by Definition 2.2 and Proposition 3.1, we have that $(0, \lambda)$ does not belong to $\sigma_{\delta,1}((y, x), H)$ if and only if $(0, \lambda)$ satisfies the following conditions:

- (i) $\bar{x}-\lambda: H/R(y) \rightarrow H/R(y)$ is an isomorphic map,
- (ii) $x-1-\lambda: Ker(y) \rightarrow Ker(y)$ is surjective.

Then, it is obvious that $\sigma_{\delta,1}((y, x), H) = \{0\} \times Sp(\bar{x}, H/R(y)) \cup \{0\} \times \Pi C(x-1|_{Ker(y)})$.

We now summarize our results.

Theorem 3.2. *Let L be the complex solvable non-commutative two dimensional Lie algebra, $L = \langle y \rangle \oplus \langle x \rangle$, with Lie bracket $[x, y]^{op} = y$, which acts as right continuous linear operators on a complex Hilbert space H . If $R(y)$ is closed, then the joint spectra $Sp(L, H)$, $\sigma_{\delta,k}(L, H)$ and $\sigma_{\pi,k}(L, H)$, $k = 0, 1, 2$, in terms of the basis $\{y, x\}$ of L and the basis of L^* dual of the latter, may be characterize as follows:*

- (i) $Sp((y, x), H) = \{0\} \times Sp(x - 1, Ker(y)) \cup \{0\} \times Sp(\bar{x}, H/R(y))$,
- (ii) $\sigma_{\delta,0}((y, x), H) = \{0\} \times \Pi C(\bar{x}, H/R(y))$,
- (iii) $\sigma_{\delta,1}((y, x)) = \{0\} \times Sp(\bar{x}, H/R(y)) \cup \{0\} \times \Pi C(x - 1, Ker(y))$,
- (iv) $\sigma_{\pi,2}((y, x), H) = \{0\} \times \Pi(x - 1, Ker(y))$,
- (v) $\sigma_{\pi,1}((y, x), H) = \{0\} \times Sp(x - 1, Ker(y)) \cup \{0\} \times \Pi(\bar{x}, H/R(y))$,
- (vi) $\sigma_{\delta,2}((y, x), H) = \sigma_{\pi,0}((y, x), H) = Sp((y, x), H)$.

4. A SPECIAL CASE

As we have seen, y is a nilpotent operator. In this section we study the case $y^2 = 0$ and we obtain a more precise characterization of the joint spectrum $Sp(L, H)$.

We decompose H in the following way: $H = Ker(y) \oplus Ker(y)^\perp$. Besides, as $R(y)$ is contained in $Ker(y)$, let us consider M , the closed subspace of H defined by $M = Ker(y) \cap R(y)^\perp$. Then we have another orthogonal direct sum decomposition of H , $H = R(y) \oplus M \oplus Ker(y)^\perp$. Moreover, if we recall that $x(R(y)) \subseteq R(y)$ and $x(Ker(y)) \subseteq Ker(y)$, then we have that x and y have the following form,

$$y = \begin{pmatrix} 0 & 0 & \bar{y} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix},$$

where \bar{y} is as in Section 3 and the maps x_{ij} , $1 \leq i \leq j \leq 2$, are the restriction of x to the corresponding spaces. We now see that, in the case under consideration, $Sp(L, H)$ reduces essentially to the spectrum of x in $Ker(y)$.

Proposition 4.1. *Let L be the complex solvable non commutative two dimensional Lie algebra $L = \langle y \rangle \oplus \langle x \rangle$, with Lie bracket $[x, y]^{op} = y$, which acts as right continuous linear operators on a complex Hilbert space H . If $R(y)$ is closed and $y^2 = 0$, $Sp(L, H)$, in terms of the basis $\{y, x\}$ of L and the basis of L^* dual of the latter, may be described as follows. If x_{11} and x_{22} are the maps defined above, and if S_i , $i = 1, 2$, are the sets: $S_1 = (Sp(x_{11}, R(y)) - 1)$, and $S_2 = (Sp(x_{22}, R(y)^\perp \cap Ker(y)))$, then, we have that,*

$$Sp((y, x), H) = \{0\} \times (S_1 \cup (S_1 + 2) \cup S_2 \cup (S_2 - 1)).$$

Proof. An easy calculation shows that the relation $[x, y]^{op} = y$ is equivalent to $\bar{y}x_{33} - x_{11}\bar{y} = \bar{y}$. However, as \bar{y} is a topological homeomorphism, $x_{33} = I_{Ker(y)^\perp} + \bar{y}^{-1}x_{11}\bar{y}$. In particular, $Sp(x_{33}, Ker(y)^\perp) = Sp(x_{11}, R(y)) + 1$. Then, as $Sp(\bar{x}, H/R(y)) = Sp(x_{22}, M) \cup Sp(x_{33}, Ker(y)^\perp)$, where $M = R(y)^\perp \cap Ker(y)$, we have that $Sp(\bar{x}, H/R(y)) = (S_1 + 2) \cup S_2$.

On the other hand, it is clear that $Sp(x - 1, Ker(y)) = S_1 \cup (S_2 - 1)$. Thus, by Theorem 1, we conclude the proof. \square

Finally, we consider the case $R(y)$ closed, $y^2 = 0$, and H finite dimensional. If $r = \dim(R(y))$ and $k = \dim(Ker(y))$, let us choose a basis of $Ker(y)$ such that the first r -vectors of it are a basis of $R(y)$, and in this basis, x has an upper triangular form, with diagonal entries λ_{ii} , $1 \leq i \leq k$. Then we have the following corollary.

Corollary 4.2. *Let H , L and the operator y be as in Proposition 4.1. If H is finite dimensional and we consider a basis of $\text{Ker}(y)$ with the above conditions, $Sp(L, H)$, in terms of the basis of L and L^* considered in Proposition 4.1, is the following set,*

$$Sp((y, x), H) = \{0\} \times \{(\lambda_{ii} - 1)_{(1 \leq i \leq k)} \cup (\lambda_{ii})_{(m \leq i \leq k)} \cup (\lambda_{ii} + 1)_{(1 \leq i \leq m)}\}.$$

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Enrico Boasso

E-mail address: enrico_odisseo@yahoo.it